CONSTRUCTION OF ŁUKASIEWICZ FILTERS

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Summary The paper deals with Łukasiewicz filters, based on aggregating of weights. Some Łukasiewicz filters are constructed and conditions to be put on the values of the weights are given.

Filters in fuzzy settings have been already studied by several authors. We refer to a series of papers [1, 2, 3, 4] where so-called generalised filters were defined. In this article we will use slightly modified definition of a generalised filter from [3]. This definition has been used also in [5]. The construction of 'fuzzy' filters will be based on aggregating of weights by t-conorms. (For basic properties of t-conorms see [6].)

1. PRELIMINARIES

Definition 1 *Let $X \neq \emptyset$ be a given set. Then $F \subseteq 2^X$ is said to be a filter on X if and only if the following hold*

1. $X \in F, \emptyset \notin F$
2. $A, B \in F \Rightarrow A \cap B \in F$
3. *If* $A \in F$ *and* $A \subseteq B$ *then* $B \in F$.

Example 1 (Filters) Let us take the set N of all natural numbers. Then we have the basic type of a filter on *N:*

$F \subset 2^N$ will have the following properties:

$$A \in F \text{ implies } A \text{ is infinite} \tag{1}$$

$$\forall A \subset N:\ A \in F \text{ or } N \setminus A \in F \tag{2}$$

This is a so-called nontrivial ultrafilter, mainly used in set-theoretical constructions.

Another example of a filter on N is the following: $A \in F \Leftrightarrow X \setminus A$ is finite.

In set-theoretical constructions important filters are those on the set N of all natural numbers which enable to get a new 'quality'. E.g., they make possible to enlarge the set R of reals into the set R^* of 'hyperreals', i.e. containing also infinitesimals (numbers which are positive, but less than any positive real). When using filters on sets of higher cardinality, we get also a higher cardinality of the enlarged universe, but from the qualitative point of view we get the same result. This is the reason why, in this paper, we will study 'fuzzy filters' on countable sets, only.

Definition 2 *Let T_L be the Łukasiewicz t-norm and $X \neq \emptyset$ be given (at most) countable set. A function $\Phi\colon 2^X \to [0,1]$ is said to be a T_L-filter if and only if the following hold*

1. $\Phi(X) = 1$, $\Phi(\emptyset) = 0$
2. *all* $A, B \subseteq X$ *yield*
$$\Phi(A \cap B) \geq T_L(\Phi(A), \Phi(B))$$
3. *all* $A, B \subseteq X$ *yield*
$$A \subseteq B \Rightarrow \Phi(A) \leq \Phi(B).$$

2. SPECIAL TYPES OF ŁUKASIEWICZ FILTERS

First, let us give the basic notation:

* $X = \{x_1, x_2, \ldots x_n, \ldots\}$ will denote an at most countable set,

* $W = \{w_1, w_2, \ldots, w_n, \ldots\}$ will denote the set of weights (i.e. $w_i \in [0,1]$ for all $w_i \in W$) assigned to the corresponding elements of *X,*

* for an arbitrary left-continuous t-conorm S we denote $S_j(w_{i_j})_j = S(\ldots S(w_{i_1}, w_{i_2}), w_{i_3}), \ldots)$.

There are special Łukasiewicz z filters, defined by the set of weights W and some t-conorm S (for more details on these filters see [1]). The definition is as follows:

Definition 3 *Let S be some left-continuous t-conorm. Denote $A = \{x_{i_j}\}_j \subseteq X$. (Remind that X is an at most countable set.) Then $\Phi_S\colon 2^X \to [0,1]$ defined by*

$$\Phi_S(A) = \begin{cases} 0, & \text{if } A = \emptyset \\ 1, & \text{if } A = X \\ S_j(w_{i_j})_j, & \text{otherwise,} \end{cases}$$

is said to be an (T_L, S)*-filter on X if it is a Łukasiewicz filter.*

In the whole paper $\Phi_S\colon 2^X \to [0,1]$ will denote the (T_L, S) filter defined by Definition 3.

The following theorems show special conditions to be put on the weighting sequence in order to get (T_L ,S)-filters using some particular t-conorms. The conditions we get applying property 2 from Definition 2 which in fact says that the membership degree of disjoint sets must not exceed 1.

Theorem 1 Φ_L is the (T_L,S_L)-filter, if for the weighting sequence the following holds

$$\sum w_i \le 1.$$

Theorem 2 Φ_M is the (T_L ,S_M)-filter, if for the weighting sequence the following holds

$$(\forall i \ne j)(w_i + w_j \le 1).$$

Theorem 3 Φ_D is the (T_L ,S_D)-filter, iff there exist at most two non-zero weights w_i, w_j $(i \ne j)$ such that $w_i + w_j \le 1$ and all other weights are equal to zero.

Next lemma contains a partial result concerning the (T_L,S_P)-filters:

Lemma 1 Let X be a finite set with an even number of elements and let the weighting sequence be constant. Then Φ_P is the (T_L ,S_P)-filter iff $w \le 1 - 2^{-2/n}$.

Example 2 Let $X = \{a,b,c,d\}$. Then:

a) Put weights of particular points as follows:
$w_a = 0{,}5, w_b = 0{,}5, w_c = 0, w_d = 0$
then the (T_L ,S_D) -filter can be constructed, but also (T_L,S_M), (T_L ,S_L) and (T_L ,S_P) -filters.

b) Put weights of particular points as follows:
$w_a = 0{,}25, w_b = 0{,}25, w_c = 0{,}25, w_d = 0{,}25$
then the (T_L ,S_L) -filter can be constructed, but also the (T_L ,S_M), (T_L ,S_P) -filters.

c) Put weights of particular points as follows:
$w_a = 0{,}29, w_b = 0{,}29, w_c = 0{,}29, w_d = 0{,}29$
then the (T_L ,S_P) -filter can be constructed, but also the (T_L ,S_M)-filter.

d) Put weights of particular points as follows:
$w_a = 0{,}5, w_b = 0{,}5, w_c = 0{,}5, w_d = 0{,}5$
then the (T_L ,S_M)-filter can be constructed and no other one.

Lemma 2 Let us have a weighting sequence $W : X \to [0,1]$, and let S, S' be left-continuous t-conorms such that $S' \le S$. Then if $\Phi_{(S,\ w)}$ is a (T_L, S)- filter, then also a (T_L, S')-filter $\Phi_{(S',\ w)}$ can be constructed.

Comment 1 In all the examples and assertions we have constructed (T_L,S)–filters, only. However, if it is possible to order the weights (which is always the case if $S_L \le S$, due to Theorem 1 and Lemma 1) then we may use also other operators then just t-conorms to construct Łukasiewicz filters. However, when relaxing the associativity property, we must keep in mind the order in which we have to aggregate the weights. We may relax even the condition $S(0,0) = 0$ to be able to distinguish between the empty set and a set containing elements of 0 weights.

Example 3 Here we present two different operators, O_1 and O_2 , which are neither associative nor $O_i(0,0) = 0$ hold for i =1,2. They both are bounded from below by the Łukasiewicz t-conorm S_L. The first operator, O_1 (Fig. 1), is piecewise linear and constant in the neighbourhood of $(0,0)$. This operator distinguishes the empty set from a set containing elements of 0 weights. However, it does not distinguish sets of different cardinalities from each other, containing elements having just 0 weights.

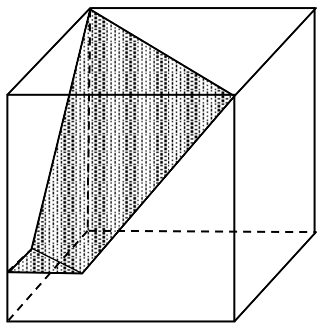

Fig. 1. Piecewise linear operator O_1

Operator O_2 (Fig. 2) is not piecewise linear in the triangle $x + y \le 1$. This operator also distinguishes the empty set from a set containing elements of 0 weights. But, unlike the operator O_1, it distinguishes sets of different cardinalities from each other, containing elements having just 0 weights.

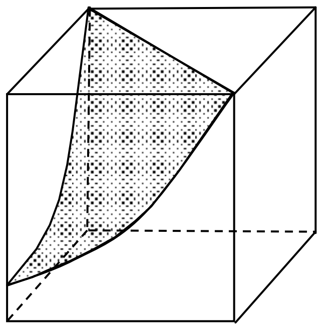

Fig. 2. Not piecewise linear operator O_2

Comment 2 However, we must always keep in mind that each operator, we use to aggregate the weights, puts some conditions on the weighting sequence to get a Łukasiewicz filter.

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